

Noncommutative phase spaces on Aristotle group

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ABSTRACT

We realize noncommutative phase spaces as coadjoint orbits of extensions of the Aristotle group in a two-dimensional space. Through these constructions the momenta of the phase spaces do not commute due to the presence of a naturally introduced magnetic field. These cases correspond to the minimal coupling of the momentum with a magnetic potential.

Keywords: noncommutative phase space, coadjoint orbits, noncentral extension, symplectic realizations, magnetic fields

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1. INTRODUCTION

Classical electromagnetic interaction can be introduced through the modified symplectic form $\sigma = dp_i \wedge dq^i + (1/2)F_{ij}dq^i \wedge dq^j$.¹⁻³ This has been initiated by J.M. Souriau³ in the seventies: one of his important theorems says in fact that when a symmetry group G acts on a phase space transitively, then the latter is a coadjoint orbit of G , endowed with its canonical symplectic form. The original applications that Souriau presented in his book³ concern both the Poincaré and the Galilei groups. Recently many authors,⁴⁻⁸ ... generalized the modification of the symplectic form by introducing the so-called dual magnetic field G by considering $\sigma = dp_i \wedge dq^i + (1/2)F_{ij}dq^i \wedge dq^j + (1/2)G^{ij}dp_i \wedge dp_j$. The fields F and G are responsible of the noncommutativity respectively of momenta and positions. Noncommutative phase spaces are then defined as spaces on which coordinates satisfy the relations:

$$\{q^i, q^j\} = G^{ij}, \quad \{q^i, p_j\} = \delta_j^i, \quad \{p_i, p_j\} = F_{ij},$$

where δ_j^i is a unit matrix, whereas G^{ij} and F_{ij} are functions of positions and momenta. Moreover the physical dimensions of G^{ij} and F_{ij} are respectively $M^{-1}T$ and MT^{-1} , M representing a mass while T represents a time.

In more recent times, Souriau's ideas were later extended to other groups. In Duval et al.⁹ for example, a classical "photon" model was constructed, based entirely on the Euclidian group $E(3)$. As the latter is simultaneously a subgroup of both the Poincaré and the Galilei groups, hence the "euclidian photon" constructed by Souriau's orbit method is indeed a reduction of both the relativistic and the nonrelativistic massless models as presented by Souriau.³ There is an intermediate group between the Euclidian and the Galilei groups dubbed, again by Souriau,¹⁰ the Aristotle group: it also contains time translations but not boosts.

This work is precisely to study the classical dynamical systems associated with this intermediate group. We use Souriau's method also called coadjoint orbit method to construct phase spaces endowed with modified symplectic structure on the Aristotle group. Explicitly, we demonstrate that such deformed objects can be generated in the framework of noncentrally extended Aristotle algebra as well as in the framework of its corresponding central extension. The obtained in such a way phase spaces do not commute in momentum sector due to the presence of a naturally introduced magnetic field. In other words, the obtained cases correspond to the minimal coupling of the momentum with a magnetic potential.

Note that there has been other more recent works about a similar construction starting with the centrally extended "anisotropic Newton-Hooke" groups¹¹ and with the noncentrally extended of both Para-Galilei and Galilei groups¹² in a two-dimensional space.

The paper is organized as follows. In section two, we give symplectic realizations of the Aristotle group in two-dimensional space using its first and second central extensions. In the third section, we realize symplectically both the noncentrally extended Aristotle Lie group and its central extension counterpart. As the coadjoint orbit construction has not been carried through this Lie group before, physical interpretations of new generators of the extended corresponding Lie algebras are also given.

2. FIRST AND SECOND CENTRAL EXTENSIONS OF THE ARISTOTLE GROUP

It is well known that a free dynamical system is a geometric object for the Aristotle group¹⁰ which is the group of both Euclidean displacements and time translations. Explicitly, the Aristotle group $A(2)$ in a two-dimensional space is a Lie group whose multiplication law is given by

$$(\theta, \vec{x}, t)(\theta', \vec{x}', t') = (\theta + \theta', R(\theta)\vec{x}' + \vec{x}, t + t'), \quad (1)$$

where \vec{x} is a space translation vector, t is a time translation parameter and θ is a rotation parameter.

Its Lie algebra \mathcal{A} is then generated by the left invariant vector fields

$$J = \frac{\partial}{\partial \theta}, \quad \vec{P} = R(-\theta) \frac{\partial}{\partial \vec{x}}, \quad H = \frac{\partial}{\partial t}$$

such that the only nontrivial Lie brackets are

$$[J, P_i] = P_j \epsilon_{ij}, \quad i, j = 1, 2. \quad (2)$$

The multiplication law (1) implies that the element g of this group can be written as:

$$g = \exp(\vec{x}\vec{p} + tH)\exp(\theta). \quad (3)$$

2.1 First central extension of $A(2)$

From the relation $\exp(2\pi i)H\exp(-2\pi i) = H$ and by use of the standard methods,¹³⁻¹⁷ we obtain the following nontrivial Lie brackets for the first central extension $\hat{\mathcal{A}}$ of \mathcal{A} (2)

$$[J, P_i] = P_i \varepsilon_i^j, \quad [P_i, P_j] = \frac{1}{r^2} S \varepsilon_{ij}, \quad i, j = 1, 2, \quad (4)$$

where S generates the center of $\hat{\mathcal{A}}$ while r is a constant whose dimension is a length.

Let g be given by (3) and $\hat{g} = \exp(\varphi S)g$ be the corresponding element in the connected Lie group associated to the extended Lie algebra $\hat{\mathcal{A}}$. By use of the Baker-Campbell-Hausdorff formulae¹⁸ and by identifying \hat{g} with $(\varphi, \theta, \vec{x}, t)$, we find that the multiplication law of the connected extended Lie group is:

$$(\varphi, \theta, \vec{x}, t)(\varphi', \theta', \vec{x}', t') = \left(\varphi' + \frac{1}{2r^2} R(-\theta)\vec{x} \times \vec{x}' + \varphi, \theta + \theta', R(\theta)\vec{x}' + \vec{x}, t + t' \right)$$

or equivalently

$$(\alpha, g)(\alpha', g') = (\alpha + \alpha' + c(g, g'), gg'),$$

where $c(g, g') = (1/2r^2)R(-\theta)\vec{x} \times \vec{x}'$ is a two-cocycle and gg' is the multiplication law (1).

The adjoint action $Ad_g(\delta\hat{g}) = g(\delta\hat{g})g^{-1}$ of A on the Lie algebra $\hat{\mathcal{A}}$ is given by:

$$Ad_{(\theta, \vec{x}, t)}(\delta\varphi, \delta\theta, \delta\vec{x}, \delta t) = \left(\delta\varphi + \frac{1}{r^2} R(-\theta)\vec{x} \times \delta\vec{x} - \frac{1}{2r^2} \vec{x}^2 \delta\theta, \delta\theta, R(\theta)\delta\vec{x} + \varepsilon(\vec{x})\delta\theta, \delta t \right)$$

with

$$\varepsilon(\vec{x}) = \begin{pmatrix} 0 & x^2 \\ -x^1 & 0 \end{pmatrix}. \quad (5)$$

If the duality between the extended Lie algebra and its dual is given by the action $j\delta\theta + \vec{p} \cdot \delta\vec{x} + l\delta\varphi + E\delta t$, where \vec{p} is a linear momentum, l is an action, j is an angular momentum while E is an energy, then the coadjoint action of the Aristotle Lie group is

$$Ad_{(\vec{x}, t, \theta)}^*(j, \vec{p}, l, E) = \left(j + \frac{m\omega}{2} (\vec{x}^2) + \vec{x} \times R(\theta)\vec{p}, R(\theta)\vec{p} - m\omega\varepsilon(\vec{x}), l, E \right),$$

where we have used the "wave-particle duality" $l\omega = mc^2$ and the relation $c = \omega r$ linking the velocity c , the frequency ω and the universe radius r .

The Kirillov form in the basis (J, P_1, P_2, H, S) is

$$K(a) = \begin{pmatrix} 0 & p_2 & -p_1 & 0 & 0 \\ -p_2 & 0 & m\omega & 0 & 0 \\ p_1 & -m\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The coadjoint orbit of the central extended Lie group on the dual of its Lie algebra is characterized by the two trivial invariants l and E , and a nontrivial invariant

$$s = j + \frac{p^2}{2m\omega} + \frac{m\omega q^2}{2}, \quad (6)$$

where p and q are defined by

$$q = -\frac{p_2}{m\omega}, \quad p = p_1. \quad (7)$$

Let us denote by $\mathcal{O}_{(S,I,E)}$ the maximal coadjoint orbit of the Aristotle group $A(2)$ on the dual of its central extended Lie algebra.

The restriction $\Omega = (\Omega_{ob})$ of the Kirillov form to the orbit is then

$$\Omega = \begin{pmatrix} 0 & m\omega \\ -m\omega & 0 \end{pmatrix}$$

It follows that the symplectic form is then $\sigma = dp \wedge dq$.

The symplectic realization of the Aristotle Lie group is

$$D_{(\theta, \vec{x}, t)}(p, q) = \left(\cos \theta p + m\omega q \sin \theta - m\omega x^2, -\frac{p}{m\omega} \sin \theta + q \cos \theta - x^1 \right).$$

The Poisson bracket corresponding to this symplectic structure is then the canonical one and the time translation subgroup acts trivially on the orbit.

To overcome this fact, let us study the symplectic realization of the second central extended Aristotle Lie group.

2.2. Second central extension

By using standard methods, we have that the second central extension of Aristotle Lie algebra in two-dimensional space is generated by: J, P_1, P_2, H, S, N satisfying the nontrivial Lie brackets:

$$[J, P_i] = P_j \varepsilon_{ij}^j, \quad [P_i, P_j] = \frac{1}{r^2} S \varepsilon_{ij}, \quad [S, H] = \omega N.$$

The multiplication law of this extended Lie group is given by:

$$\begin{aligned} &(\psi, \varphi, \theta, \vec{x}, t)(\psi', \varphi', \theta', \vec{x}', t') \\ &= \left(\psi + \psi' - \omega t \varphi', \varphi' + \frac{1}{2r^2} R(-\theta) \vec{x} \times \vec{x}' + \varphi, \theta + \theta', R(\theta) \vec{x}' + \vec{x}, t + t' \right). \end{aligned}$$

The adjoint action of \hat{A} on its extended Lie algebra is explicitly given by

$$\begin{aligned} &Ad_{(\varphi, \theta, \vec{x}, t)}(\delta\psi, \delta\varphi, \delta\theta, \delta\vec{x}, \delta t) \\ &= \left(\delta\psi - \omega t \delta\varphi + \omega \varphi \delta t, \delta\varphi + \frac{1}{r^2} R(-\theta) \vec{x} \times \delta\vec{x} - \frac{1}{2r^2} \vec{x}^2 \delta\theta, \delta\theta, R(\theta) \delta\vec{x} + \varepsilon(\vec{x}) \delta\theta, \delta t \right), \end{aligned}$$

where $\varepsilon(\vec{x})$ is given by the relation (5).

If the duality between the extended Lie algebra and its dual is given by the action $j\delta\theta + \vec{p} \cdot \delta\vec{x} + E\delta t + l\delta\varphi + h\delta\psi$, then the coadjoint action of the extended Aristotle Lie group is such that

$$\begin{aligned} &Ad_{(\varphi, \vec{x}, t, \theta)}^*(j, \vec{p}, E, l, h) \\ &= \left(j + \vec{x} \times R(\theta) \vec{p} - \frac{l + h\omega t}{2r^2} \vec{x}^2, R(\theta) \vec{p} + \frac{l}{r^2} \varepsilon(\vec{x}) + \frac{h}{r^2} \omega t \varepsilon(\vec{x}), E - h\omega\varphi, l + h\omega t, h \right) \quad (8) \end{aligned}$$

meaning that h is a trivial invariant.

In the basis (J, P_1, P_2, H, S, N) , the Kirillov form is

$$K(a) = \begin{pmatrix} 0 & p_2 & -p_1 & 0 & 0 & 0 \\ -p_2 & 0 & m\omega & 0 & 0 & 0 \\ p_1 & -m\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -h\omega & 0 \\ 0 & 0 & 0 & h\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

The coadjoint orbit of \hat{A} on the dual $\hat{\mathcal{A}}^*$ of the second central extension Lie algebra is characterized by the trivial invariant h and by the nontrivial invariant given by relation (6).

The maximal coadjoint orbit is quadri-dimensional and is denoted by $\mathcal{O}_{(h,S)}$.

The restriction $\Omega = (\Omega_{ab})$ of the Kirillov form (10) to the orbit is then

$$\Omega = \begin{pmatrix} 0 & m\omega & 0 & 0 \\ -m\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -h\omega \\ 0 & 0 & h\omega & 0 \end{pmatrix}.$$

It follows that the symplectic form is in this case given by

$$\sigma = dp \wedge dq + d\alpha \wedge dl,$$

where

$$\alpha = \frac{E}{h\omega}.$$

From the relations (8), we get that the symplectic realizations of the extended group on its maximal coadjoint orbit $(p', q', l', \alpha') = D_{(\varphi, \theta, x_1, x_2, t)}(p, q, l, \alpha)$ is

$$p' = \cos \theta p - m\omega \sin \theta q - m\omega x^2 + \frac{h}{r^2} \omega x^2 t, \quad l' = l + h\omega t$$

$$q' = \frac{1}{m\omega} \sin \theta p + \cos \theta q - x^1 - \frac{h}{mr^2} x^1 t, \quad \alpha' = \alpha + \varphi.$$

The Poisson bracket of two functions f_1 and f_2 on the orbit corresponding to the above symplectic form is

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial q} - \frac{\partial f_1}{\partial q} \frac{\partial f_2}{\partial p} + \frac{\partial f_1}{\partial l} \frac{\partial f_2}{\partial \alpha} - \frac{\partial f_1}{\partial \alpha} \frac{\partial f_2}{\partial l}.$$

It follows that

$$\{p, q\} = 1, \quad \{l, \alpha\} = 1$$

the other Poisson brackets being trivial.

The equations of motion are then

$$\frac{dp}{dt} = 0, \quad \frac{dl}{dt} = h\omega, \quad \frac{dq}{dt} = 0, \quad \frac{d\alpha}{dt} = 0.$$

In this case, the coadjoint orbit is a direct product of two 2-dimensional phase spaces $R^2 = \{(p, q)\}$ and $R^2 = \{(l, \alpha)\}$. Note that α is a dimensionless quantity. For its particular value $\alpha = 1/2\pi$, the

energy E is given by

$$E = \hbar\omega,$$

where $\hbar = 2\pi\hbar$, relation analogue to that of Quantum Mechanics.

With the second central extension of Aristotle group in two dimensional space, we have then also realized a phase space with commuting coordinates (canonical case). Moreover, the position and the linear momentum do not depend on time.

We prove, in the following section, that noncommutative phase spaces can be obtained by considering the noncentral extension of the two-dimensional Aristotle group.

3. NONCENTRAL EXTENSION OF ARISTOTLE GROUP

In the previous section, we find that one can not construct noncommutative phase spaces by coadjoint orbit method on the first and second central extensions of the Aristotle group because symplectic structures obtained are canonical which means that positions commute as well as momenta.

In this section, we see that this construction is possible when we consider a noncentral extension of this Lie group.

3.1. Noncentrally extended group and its maximal coadjoint orbit

Let $\hat{\mathcal{A}}_1$ be the noncentrally extended Aristotle Lie algebra satisfying the non trivial Lie brackets

$$[J, P_i] = P_j \epsilon_{ij}^k, \quad [J, F_i] = F_k \epsilon_{ij}^k, \quad [P_i, P_j] = \frac{1}{r^2} S \epsilon_{ij}, \quad [P_i, H] = F_i, \quad i, j = 1, 2. \quad (10)$$

If $\hat{g} = \exp(\varphi S + \vec{\eta} \vec{F}) \exp(\vec{x} \vec{P}) \exp(\theta J) \exp(tH)$ is the general element of the connected extended Aristotle group, we verify that the corresponding multiplication law is

$$(\varphi', \theta', \vec{\eta}', \vec{x}', t') = (\varphi, \theta, \vec{\eta}, \vec{x}, t)(\varphi', \theta', \vec{\eta}', \vec{x}', t')$$

with

$$\begin{aligned} \varphi'' &= \varphi' + \frac{1}{2r^2} R(-\theta) \vec{x} \times \vec{x}' + \varphi, \quad \vec{\eta}'' = R(\theta) \vec{\eta}' - R(\theta) \vec{x}' t + \vec{\eta} \\ \vec{x}'' &= R(\theta) \vec{x}' + \vec{x}, \quad \theta'' = \theta' + \theta, \quad t'' = t' + t. \end{aligned}$$

It follows that the adjoint action of the noncentral extended Aristotle group on its Lie algebra is such that

$$\begin{aligned} \partial \varepsilon \delta \theta' &= \delta \theta, \quad \delta t' = \delta t, \quad \delta \vec{x}' = R(\theta) \delta \vec{x} + \varepsilon(\vec{x}) \delta \theta \\ \delta \vec{\eta}' &= R(\theta) \delta \vec{\eta} + \varepsilon(\vec{\eta}) \delta \theta - R(\theta) \delta \vec{x} t + \vec{x} \delta t \\ \delta \varphi' &= \delta \varphi + \frac{1}{r^2} R(\theta) \vec{x} \times \delta \vec{x} - \frac{\vec{x}^2}{2r^2} \delta \theta. \end{aligned}$$

If the duality between the extended Lie algebra and its dual is given by the action $j \delta \theta + \vec{f} \cdot \delta \vec{\eta} + \vec{p} \cdot \delta \vec{x} + h \delta \varphi + E \delta t$, then the coadjoint action is such that $h' = h$ and

$$\vec{f}' = R(\theta) \vec{f}, \quad \vec{p}' = R(\theta) \vec{p} + R(\theta) \vec{f} t + \frac{h}{r^2} \varepsilon(\vec{x}) \quad (11)$$

$$j' = j + \vec{x} \times R(\theta) \vec{p} + \vec{\eta} \times R(\theta) \vec{f} - \frac{h}{r^2} \vec{x}^2 \quad (12)$$

$$E' = E - \vec{x} \cdot R(\theta) \vec{f}.$$

The coadjoint orbits denoted by $\mathcal{O}_{(h,f,U)}$ are characterized by the trivial invariant h and by two nontrivial invariants f and U given by:

$$f = \|\vec{f}\|, \quad U = E + \frac{1}{m\omega} (\vec{p} \times \vec{f}),$$

where the wave-particle duality and the relation $c = \omega r$ have been used.

Let $f_1 = f \cos \phi$, $f_2 = f \sin \phi$. The inverse of the restriction of the Kirillov form on the coadjoint orbit in the basis (J, F_1, P_1, P_2) is

$$\Omega^{-1} = \frac{1}{m\omega f \sin \phi} \begin{pmatrix} 0 & -m\omega & 0 & 0 \\ m\omega & 0 & p_1 & p_2 \\ 0 & -p_1 & 0 & -f \sin \phi \\ 0 & -p_2 & f \sin \phi & 0 \end{pmatrix}.$$

The symplectic form on the orbit is then

$$\sigma = dj \wedge d\phi + dp \wedge dq + \frac{p}{m\omega} dp \wedge d\phi + m\omega q dq \wedge d\phi, \quad (13)$$

where p_1 and q are given by (7).

The invariant U can be written as $U = E + v(p \sin \phi + m\omega q \cos \phi)$ where $v = f/m\omega$ is a velocity.

The Poisson bracket of two functions g_1 and g_2 corresponding to the symplectic form (13) is given by

$$\begin{aligned} \{g_1, g_2\} &= \frac{\partial g_1}{\partial p} \frac{\partial g_2}{\partial q} - \frac{\partial g_1}{\partial q} \frac{\partial g_2}{\partial p} + \frac{\partial g_1}{\partial j} \frac{\partial g_2}{\partial \phi} - \frac{\partial g_1}{\partial \phi} \frac{\partial g_2}{\partial j} \\ &+ m\omega q \left(\frac{\partial g_1}{\partial j} \frac{\partial g_2}{\partial p} - \frac{\partial g_1}{\partial p} \frac{\partial g_2}{\partial j} \right) - \frac{p}{m\omega} \left(\frac{\partial g_1}{\partial j} \frac{\partial g_2}{\partial q} - \frac{\partial g_1}{\partial q} \frac{\partial g_2}{\partial j} \right). \end{aligned}$$

We then have the following non trivial Poisson brackets within the coordinates on the maximal coadjoint orbit:

$$\{j, p\} = m\omega q, \quad \{\phi, q\} = 0 \quad (14)$$

$$\{j, \phi\} = 1, \quad \{p, q\} = 1 \quad (15)$$

$$\{j, q\} = -\frac{p}{m\omega}, \quad \{\phi, p\} = 0. \quad (16)$$

The relations (14) mean that momenta (j, p) do not commute but the configurations coordinates (ϕ, q) commute, the relations (15) mean that j is conjugated to ϕ and that p is conjugated to q , the relations (16) mean that j do not commute with q while p commute with ϕ .

Let the symplectic realization of the extended Aristotle Lie group on its coadjoint orbit be given by $(j', \phi', p', q') = L_{(\theta, \eta^1, \eta^2, x^1, x^2, t)}(j, \phi, p, q)$. By using relations (11) to (12), we obtain

$$j' = j + p(\sin \theta x^1 - \cos \theta x^2) - m\omega q(\cos \theta x^1 + \sin \theta x^2) + \vec{\eta} \times R(\theta) \vec{f} - m\omega \vec{x}^2$$

$$p' = \cos \theta p + m\omega \sin \theta q - m\omega x^2, \quad q' = -\frac{1}{m\omega} \sin \theta p + \cos \theta q + x^1, \quad \phi' = \phi + \theta.$$

It follows that $L_{(\theta, \eta^1, \eta^2, x^1, x^2, t)}(j, \phi, p, q) = (j, \phi, p, q)$ meaning that all the coordinates j, ϕ, p and q on the maximal coadjoint orbit are constant with respect to the time t . To overcome this situation, let us consider the central extension of the noncentrally extended Aristotle group.

But first note also that the symplectic form (13) can be written in the canonical way as

$$\sigma = dH \wedge d\tau + dp \wedge dq,$$

where $H = j\omega + (p^2/2m) + (m\omega^2 q^2/2)$ is an energy while $\tau = \alpha/\omega$ is a time.

3.2 Central extension of the noncentrally extended Aristotle group

Consider the central extension of the Lie algebra defined by (10) satisfying the non trivial Lie brackets

$$[J, P_j] = P_i \epsilon_{ij}^j, \quad [P_i, P_j] = \frac{1}{r^2} S \epsilon_{ij} \quad (17)$$

$$[J, F_j] = F_i \epsilon_{ij}^j, \quad [P_i, H] = F_i, \quad [P_i, F_j] = K \delta_{ij}.$$

We recover the Lie algebra defined by (2) when $F_i = 0, K = 0$ and $S = 0$, the Lie algebra defined by (4) when $F_i = 0, K = 0$ and the Lie algebra defined by (10) when $K = 0$. Consider now the general Lie algebra defined by (17).

Let $\hat{g} = \exp(\varphi S + \gamma K) \exp(tH) \exp(\vec{\eta}\vec{F} + \vec{x}\vec{P}) \exp(\theta)$ be the general element of the corresponding connected extended Aristotle group. By identifying \hat{g} with $(\varphi, \gamma, t, \vec{\eta}, \vec{x}, \theta)$ the multiplication law $\hat{g}'' = \hat{g}'\hat{g}'$ is such that

$$\varphi'' = \varphi' + \frac{1}{2r^2} \vec{x} \times R(-\theta) \vec{x}' + \varphi, \quad \theta'' = \theta' + \theta, \quad t'' = t + t',$$

$$\gamma'' = \gamma' + \frac{1}{2} \vec{x} \cdot R(\theta) \vec{\eta}' - \frac{1}{2} (\vec{\eta} + \vec{x}t') \cdot R(\theta) \vec{x}' + \gamma,$$

$$\vec{\eta}'' = R(\theta) \vec{\eta}' + \vec{\eta} + \vec{x}t', \quad \vec{x}'' = R(\theta) \vec{x}' + \vec{x}.$$

It follows that the adjoint action of the extended Aristotle group on its Lie algebra is such that

$$\delta\gamma' = \delta\gamma + \vec{x} \times R(\theta) \delta\vec{\eta} - \vec{\eta} \times R(\theta) \delta\vec{x} - \vec{\eta} \times \vec{x} \delta\theta + \frac{1}{2} \vec{x}^2 \delta t$$

$$\delta\vec{\eta}' = R(\theta) \delta\vec{\eta} + \varepsilon(\vec{\eta} - \vec{x}t) \delta\theta - tR(\theta) \delta\vec{x} + \vec{x} \delta t$$

$$\delta\varphi' = \delta\varphi + \frac{1}{r^2} R(-\theta) \vec{x} \times \delta\vec{x} - \frac{\vec{x}^2}{2r^2} \delta\theta$$

$$\delta\vec{x}' = R(\theta) \delta\vec{x} + \varepsilon(\vec{x}) \delta\theta, \quad \delta t' = \delta t, \quad \delta\theta' = \delta\theta,$$

where $\varepsilon(\vec{x})$ is given by the relation (5).

If the duality between the extended Lie algebra and its dual Lie algebra gives rise to the action $j\delta\theta + \vec{f} \cdot \delta\vec{\eta} + \vec{p} \cdot \delta\vec{x} + h\delta\varphi + E\delta t + k\delta\gamma$, then the coadjoint action is such that

$$h' = h, \quad k' = k \tag{18}$$

and

$$\vec{p}' = R(\theta) \vec{p} + R(\theta) \vec{f}t + k(\vec{\eta} - \vec{x}t) + \frac{h}{r^2} \varepsilon(\vec{x}), \quad \vec{f}' = R(\theta) \vec{f} - k\vec{x} \tag{19}$$

$$j' = j + \vec{x} \times R(\theta) \vec{p} + \vec{\eta} \times R(\theta) \vec{f} - \frac{h}{2r^2} \vec{x}^2$$

$$E' = E - \vec{x} \cdot R(\theta) \vec{f} + \frac{1}{2} k\vec{x}^2,$$

where \vec{p} is a linear momentum, h is an action, \vec{f} is a force, k is Hooke's constant, E is an energy and j is an angular momentum.

The coadjoint orbit is, in this case, characterized by the two trivial invariants h and k (18), and by the nontrivial invariants s and U given by:

$$s = j - \vec{p} \times \vec{q} + \frac{1}{2} m\omega\vec{q}^2, \quad U = E - \frac{1}{2} k\vec{q}^2,$$

where

$$\vec{q} = -\frac{\vec{f}}{k}. \tag{20}$$

We see that the coadjoint orbit is four-dimensional. Let us denote it by $\mathcal{O}_{(h,k,s,U)}$.

The restriction $\Omega = (\Omega_{ob})$ of the Kirillov form to the orbit is then

$$\Omega = \begin{pmatrix} 0 & m\omega & k & 0 \\ -m\omega & 0 & 0 & k \\ -k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \end{pmatrix}.$$

The modified symplectic form is explicitly given by

$$\sigma = dp_i \wedge dq^i + \frac{1}{2} m\omega \varepsilon^{ij} dq^i \wedge dq^j,$$

where \vec{q} is given by relation (20).

If $(y_a) = (p_1, p_2, q^1, q^2)$, the Poisson brackets are then explicitly given by

$$\{H, g\} = \frac{\partial H}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial g}{\partial p_i} + F_{ij} \frac{\partial H}{\partial p_i} \frac{\partial g}{\partial p_j},$$

where

$$F_{ij} = -m\omega \varepsilon_{ij}.$$

This implies that

$$\{p_i, p_j\} = F_{ij}, \{p_i, q^j\} = \delta_i^j, \{q^i, q^j\} = 0.$$

Let the symplectic realization of the extended Aristotle Lie group on its coadjoint orbit be given by $(\vec{p}', \vec{q}') = L_{(\theta, \vec{\eta}, \vec{x}, t)}(\vec{p}, \vec{q})$. By using relations (19), we have

$$\vec{p}' = R(\theta)\vec{p} - k[(R(\theta)\vec{q} + \vec{x})t - \vec{\eta}] + h\varepsilon(\vec{x}), \vec{q}' = R(\theta)\vec{q} + \vec{x}.$$

It follows that $(\vec{p}(t), \vec{q}(t)) = D_{(0,0,0,0,t)}(\vec{p}, \vec{q})$ gives rise to

$$\vec{p}(t) = \vec{p} - k\vec{q}t, \vec{q}(t) = \vec{q}.$$

The equations of motion are then

$$\frac{d\vec{p}}{dt} = -k\vec{q}, \quad \frac{d\vec{q}}{dt} = 0.$$

So with the central extension of the noncentrally extended of the two-dimensional Aristotle group, we have realized a phase space where momenta do not commute and this noncommutativity is due to presence of the magnetic field

$$F_{ij} = -m\omega \varepsilon_{ij} = -eB\varepsilon_{ij}. \tag{21}$$

Moreover, this phase space (the orbit) describes a spring submitted to a Hooke's force ($\vec{F} = -k\vec{q}$) which does not change the elongation in time.

4. CONCLUSION

In this paper, we have proved that one can not construct noncommutative phase spaces by the coadjoint orbit method with the first and the second central extensions of the two-dimensional Aristotle group because symplectic structures obtained are canonical. But by considering the noncentrally extended Aristotle group and its corresponding central extension, we have realized partially noncommutative phase spaces (only momenta do not commute). In the first case, all the phase space coordinates do not depend on the time. To overcome this situation, we have considered the central extension of the above noncentrally extended Aristotle group. The phase space obtained in the latter case describes a spring submitted to a Hooke's force ($\vec{F} = -k\vec{q}$) which does not change the elongation in time. Furthermore, the noncommutativity of momenta is measured by a term which is

associated to the naturally introduced magnetic field (21). Moreover, this case corresponds to the minimal coupling of the momentum with the magnetic potential.¹²

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